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OBSERVATION OF CHAOS IN A NONLINEAR OSCILLATOR WITH DELAY: A NUMERICAL STUDY

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The Van der Pol-Duffing's oscillator with time delay is investigated. Some examples of the different behaviour of strange attractors with the changing of delay parameters are given. For certain parameter values the motion of the oscillator is harmonic with only one frequency.

Key Words: Chaos, Strange Attractor, Bifurcation Tree, Fast Fourier Transform

1. INTRODUCTION

Irregular motion has already been mentioned in the classical works on nonlinear vibrations (Minorsky, 1962 ; Hayashi, 1964). However, no attention was given to this kind of motion. The first example of chaotic motion was presented by Lorenz (Lorenz, 1963). He investigated the problem of convection in a layer of fluid with a finite thickness, and the problem was reduced to the solution of three ordinary nonlinear differential equations. The Lorenz equations were examined by Sparrow (Sparrow, 1982), where extensive references concerning chaos were included.

After the publication of Lorenz's paper, the classical equations governing the nonlinear dynamical systems were reconsidered. Ueda found chaos in the Duffing's oscillator (Ueda, 1979), and some years later Ueda and Akamatsu presented an example of the strange attractor in the Van der Pol-Duffing's equation (Ueda, Akamatsu, 1981). Holmes discovered the strange attractor in the Duffing's oscillator with negative linear elasticity (Holmes, 1979).

At present, there are numerous examples of chaotic motion in simple physical systems in various fields of science, eg. in mechanical systems, electrical circuits, rotary magnetomechanical devices, feedback control devices, chemical systems, in lasers and optical resonators, in cellular metabolism and physiological systems and in cardiac rhytms (Holmes, Moon, 1983 ; Holden, 1986).

This work presents some different examples of behaviour of the oscillator with the change of delay parameters, value of amplification coefficient k and delay argument τ_0. The consider here the equation governing the vibration of a mechanical system with one degree of freedom, where the linear spring possesses a time delay in its action (see for instance Plaut and Hsieh, 1987). The chaotic behaviour of this oscillator was earlier considered by Awrejcewicz using an analytical approach (Awrejcewicz, 1988).

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2. ANALYSED SYSTEM

The nonlinear oscillator governed by equation

$$\ddot{x} + \alpha(1-x^2)\dot{x} + \beta x^3 = kx(t-\tau_0) + F\cos\omega t \qquad (1)$$

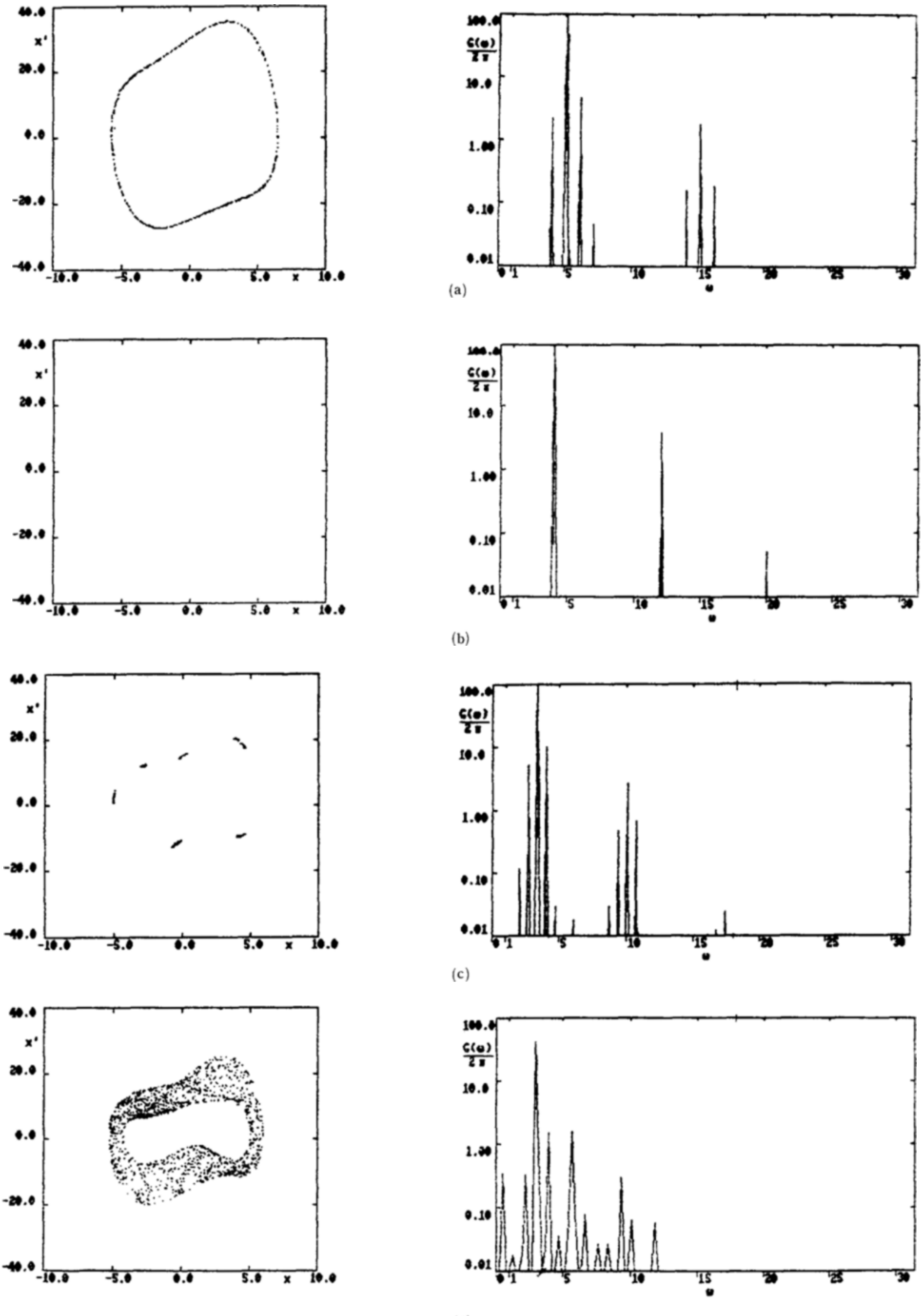
is analysed. Eq. (1) has been solved numerically with use of the modified Runge-Kutta method of the fourth order. The initial function was that, $x(t)=1.0$ for $\tau_0 \le t < 0$ and $x(t)=0$ for $\tau=0$, and $\dot{x}(t)=0$ for $-\tau_0 \le t \le 0$.

The results of computer calculations are presented as Poincare maps and frequency spectra (FFF procedure was used). The results have been recorded in the time interval $T_{min} \le t \le T_{max}$, with the calculation step 0.05. A relatively high value of T_{min} has been chosen that the transient state resulting from the introduction of the initial function would be decayed, and so allow a single trajectory to wander over the final attractor. On the other hand, the higher the value of T_{max}, the more points can be obtained in the Poincare map.

For $\alpha=0.2$, $\beta=1.0$, $F=17.0$, $\omega=4.0$ and $k=0.0$ Ueda and Akamatsu have shown that oscillator (1) has chaotic orbits. Let us now investigate the behaviour of the strange attractor associated with the changes of the delay coefficients k and τ_0.

3. A ROUTE TO CHAOS VIA SUCCESSIVE BIFURCATION OF FREQUENCY

We now observe the development of behaviour of the oscillator for $k=10.0$ with the increase of the delay argument τ_0 (other parameters are fixed). For $\tau_0=1.0$ we have obtained regular motion. For this case the Poincare map contains the points lying on the regular closed curve and the corresponding frequency spectra are discrete (Fig. 1a). For $\tau_0=1.5$ the periodic motion has appeared. This motion includes the harmonic frequency of the excited force and the third and fifth ultraharmonics. Then (for $\tau_0=1.75$) the motion becomes quasiperiodic again with the characteristic three groups of frequency. With the further increase of τ_0 (Fig. 1d) a "weak"



x'
x
$\frac{G(\omega)}{2\pi}$
ω
(a)
(b)
(c)
(d)

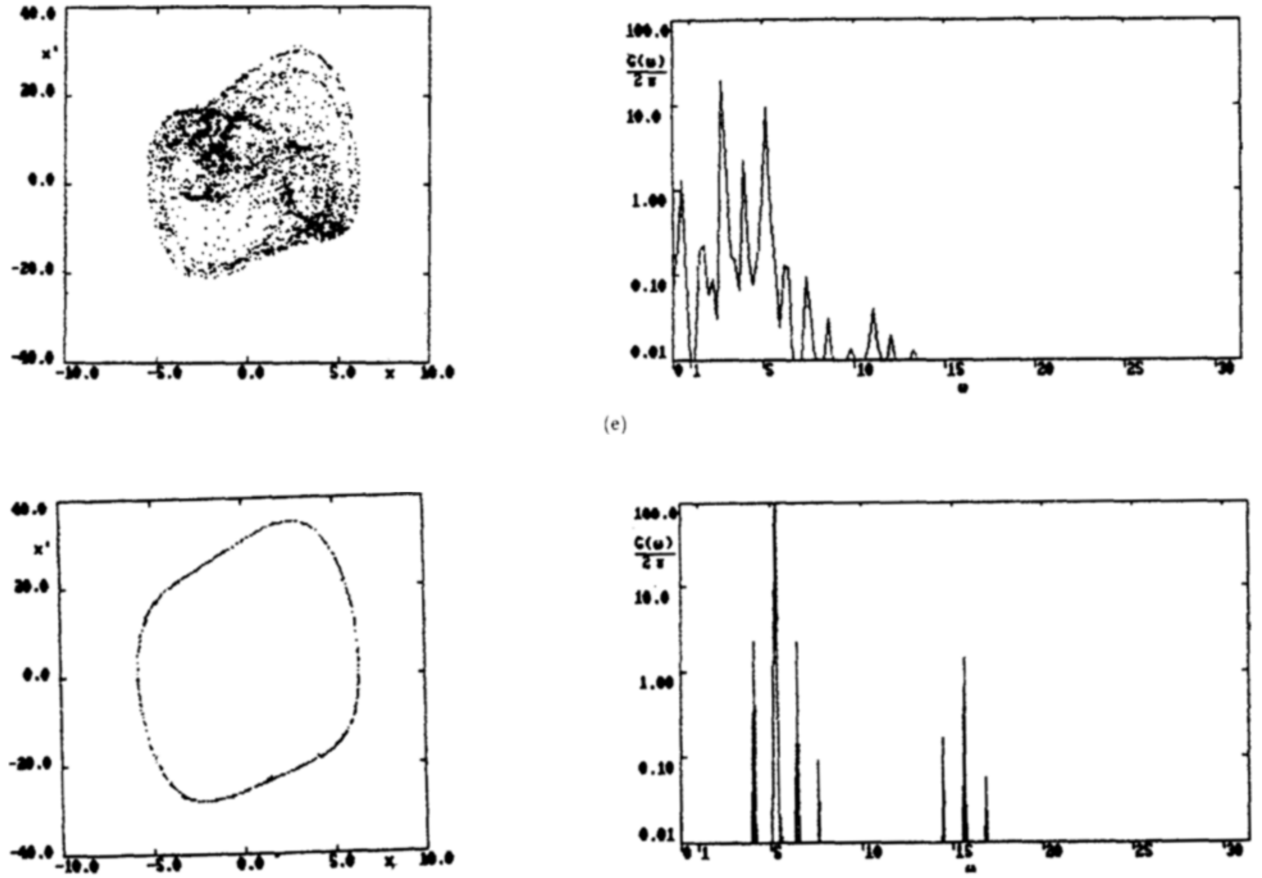


Fig. 1 Poincare maps (left) and frequency spectra (right) for $k=10.0$; ($T_{min}=250.0$, $T_{max}=3000.0$) ; a : $\tau_0=1.0$, b : $\tau_0=1.5$, c : $\tau_0=1.75$; d : $\tau_0=1.9$, e : $\tau_0=2.0$, f : $\tau_0=2.15$

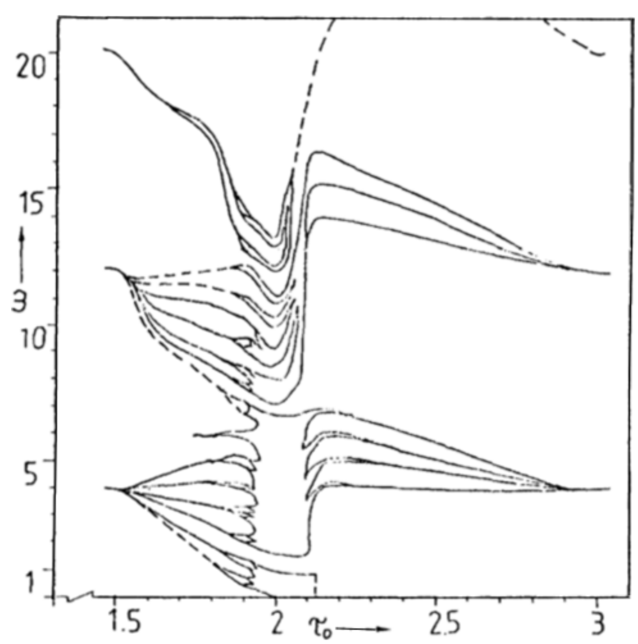


Fig. 2 Frequencies versus time delay τ_0 for the parameters as in Fig. 1

chaos is exhibited. The points on the Poincare map appear in an irregular way and the previously discrete values in the Fourier spectra tend to broaden slightly.

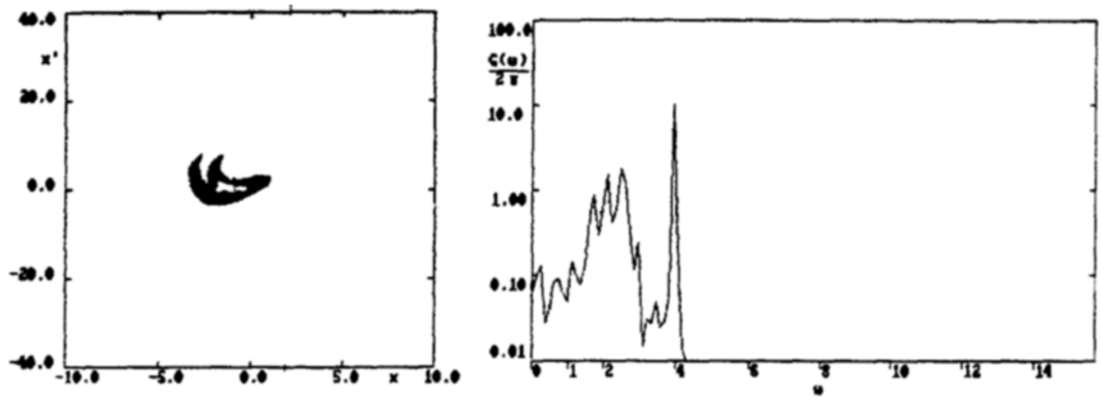
The strange attractor is then detected (Fig. 1e) with the characteristic structure of the Poincare map, the Fourier spectra having infinitely many components. The irregular motion exists however in a relatively many components. The irregular motion exists however in a relatively small interval of τ_0. For $\tau_0=2.15$ the motion is almost periodic again (Fig. 1f).

In Fig. 2 a part of the "bifurcation tree" is presented, illustrating the transition from the regular motion to chaos, and then to regular again.

In this case we have shown that a further strange attractor exists near to Ueda's for $k=0.0$. We have also demonstrated the transition from regular guasiperiodic motion via chaos to an another regular quasiperiodic motion with increasing delay argument τ_0.

4. A ROUTE FROM CHAOS TO REGULAR MOTION

Now we consider the change of the strange attractor (for fixed value $\tau_0=1.0$) with increase of amplification coefficient k. From $k=0.0$ to $k=0.1$ the strange attractor becomes chaotic (see Fig. 3a, b). In both cases presented (for $k=0.001$ and


40.0
x'
20.0
0.0
-20.0
-40.0
-10.0
-5.0
0.0
5.0
x
10.0
100.0
G(ω)/2π
10.0
1.00
0.10
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1
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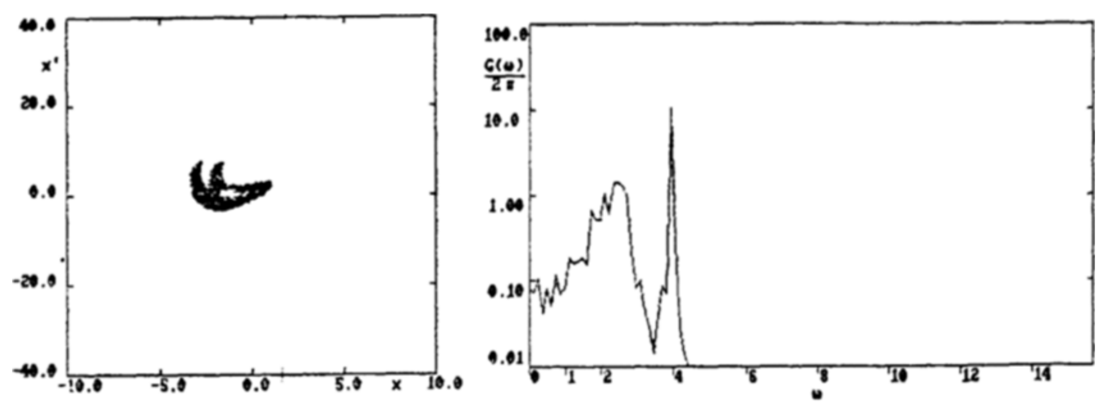
8
10
12
14
ω

(a)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x
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G(ω)/2π
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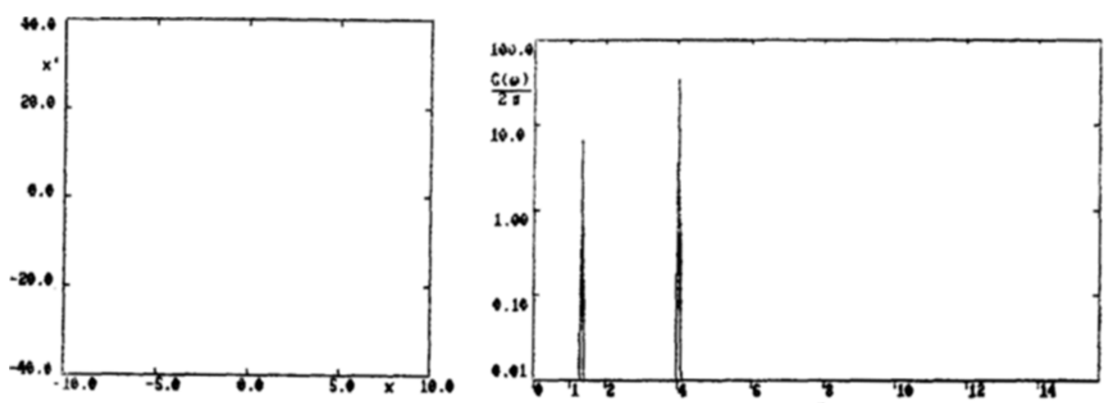
(b)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G(ω)/2π
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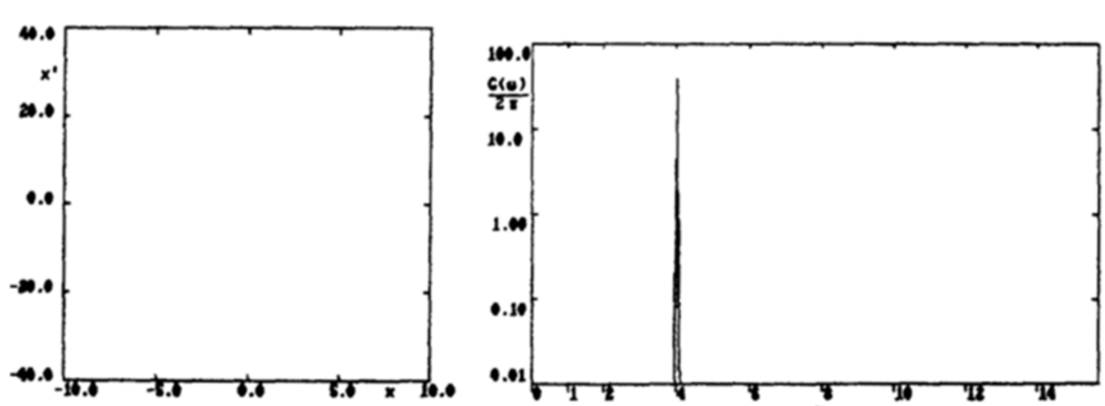
(c)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x
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100.0
G(ω)/2π
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12
14
ω

(d)

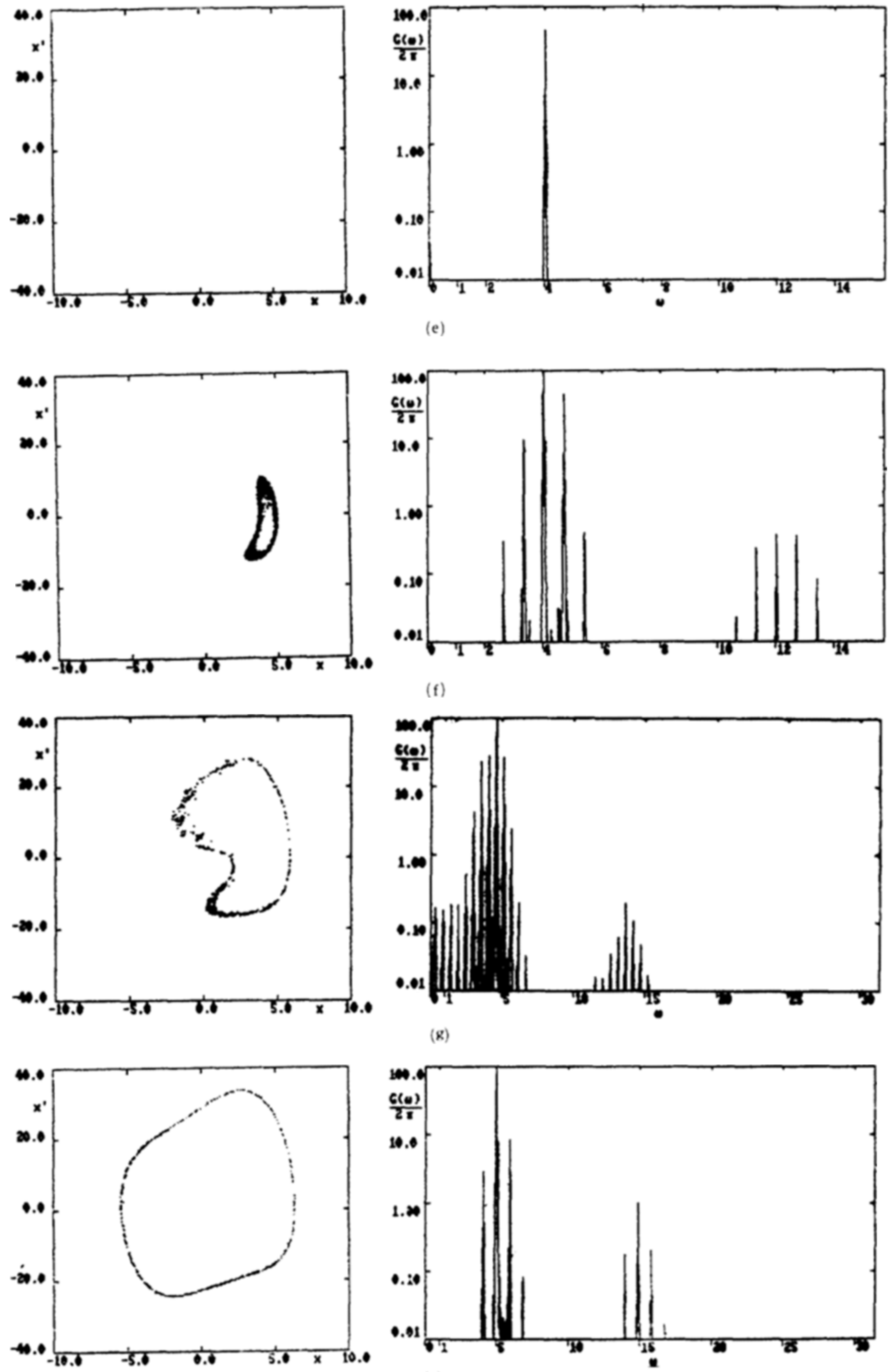


Fig. 3 Poincare maps (left) and frequency spectra (right) for $\tau_0=1.0$($T_{min}=39.3$, $T_{max}=2000.0$); a: $k=0.001$, b: $k=0.01$, c: $k=0.1$, d: $k=2.0$, e: $k=3.0$, f: $k=6.75$, g: $k=7.25$, h: $k=9.0$

$k=0.01$) the broad-band Fourier spectra lie on the left side of the fundamental frequency $\omega=4$. For $k=0.1$ the strange chaotic attractor suddenly disappears and periodic motion occurs with a fundamental frequency (i.e. corresponding to exciting force) and one subharmonic. For $k=2.0$ we present the interesting result that, even in the strong nonlinear oscillator, the periodic orbit with one frequency (equal to ω) can appear. This harmonic motion remains unchanged with the further increase of τ_0. In the interval $6.75 \leq k \leq 7.25$, attractors are observed whose orbits show very complicated motion. The question arises as to whether these two attractors presented (Fig. 3f, g) are strange non-chaotic or classical quasiperiodic attractors. According to Grebogi et. al., and Grebogi, Ott, Yorke there exist strange attractors which are non-chaotic (Grebogi et. al., 1984 and Grebogi, Ott, Yorke, 1987). These attractors possess a dimension which is not an integer, and the Liapunov exponent is not positive. We have not proved that the attractors which are shown in Fig. 3f, g are strange non-chaotic. However, when we compare the next Poincare map for $k=9$(where the quasiperiodic motion is demonstrated) the points of the attractor lie on the regular closed curve.

5. PERIODIC WINDOW WITH TWO FREQUENCIES

Let us observe the influence of changing the delay argument τ_0 for the fixed value $k=0.1$(Fig. 4). For $\tau_0=2.0$ the strange chaotic attractor appears (Fig. 4a). Chaotic orbital states remain up to $\tau_0=5.0$. For $\tau_0=5.0$ periodic motion with two frequencies has appeared. Then chaotic motion is observed again (Fig. 4d).

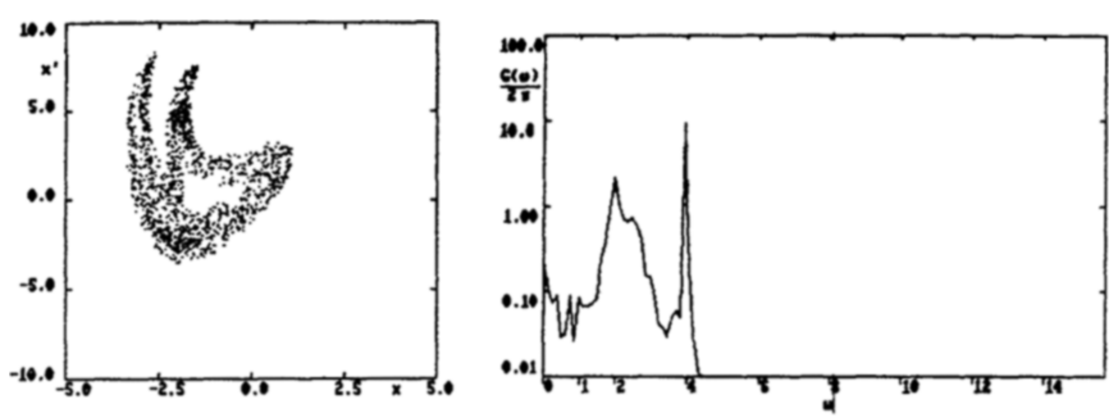
In the τ_0 interval considered here the dominant motion was chaotic. We have found however, between the two types of strange attractor, periodic motion with fundamental and one subharmonic frequency.

6. PERIODIC WINDOW WITH ONE FREQUENCY

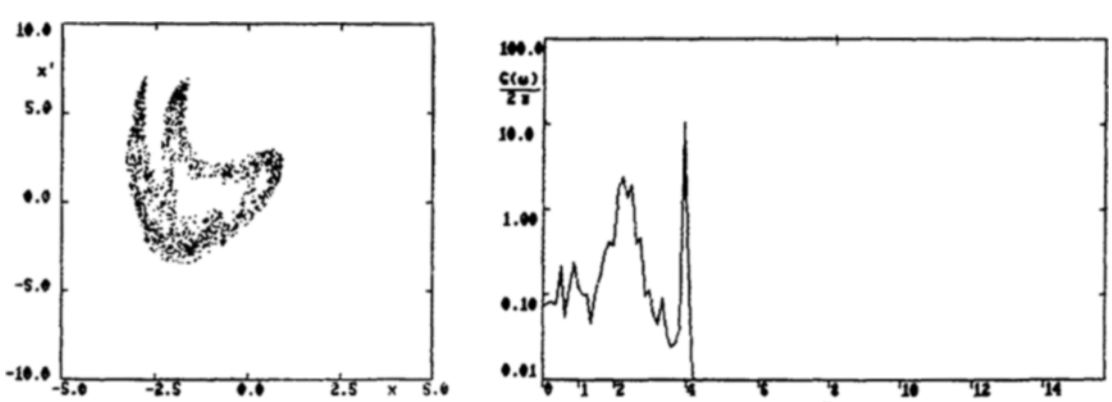
Now, our procedure will be analogous to the case described on p. 5, but the fixed value is $k=1.0$. For $\tau_0=1.0$, even though the oscillator is strongly nonlinear, the motion is harmonic with the frequency of exciting force(Fig. 5a). For $\tau_0=2.0$(Fig. 5b) chaotic orbits occur. The continuous part of the Fourier spectra lies left of $\omega=4$. With increasing τ_0, chaotic behaviour of the oscillator is more evident(Fig. 5c, d, e).

7. EVOLUTION OF THE STRANGE CHAOTIC ATTRACTORS

We have observed that for a fixed value of $k=0.001$ and with continuous variation of τ_0, the motion of the oscillator is irregular only. Some examples of the calculated results are shown in Fig. 6. Inspite of a large change in τ_0, the strange attractor evolves, but does not evolve into a regular attractor. In a general sense the four presented Poincare



(a)



(b)

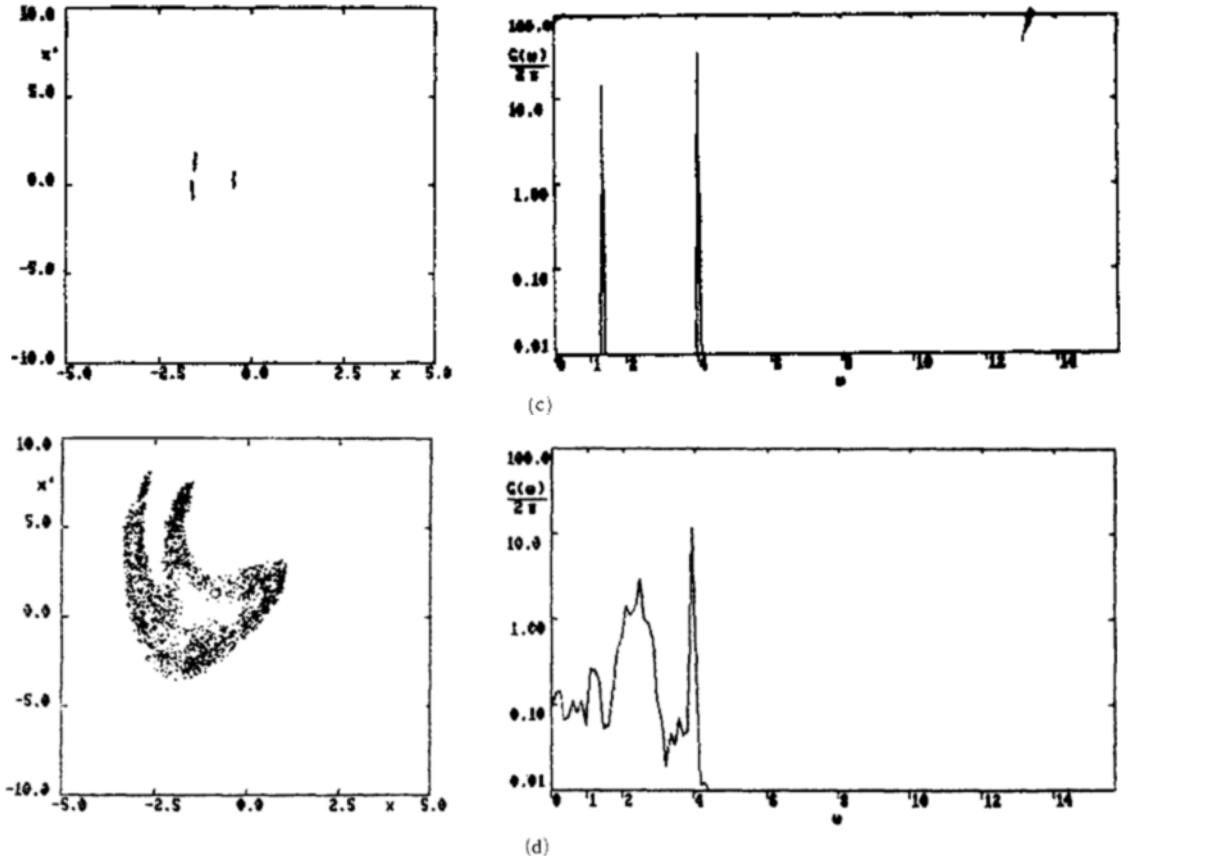
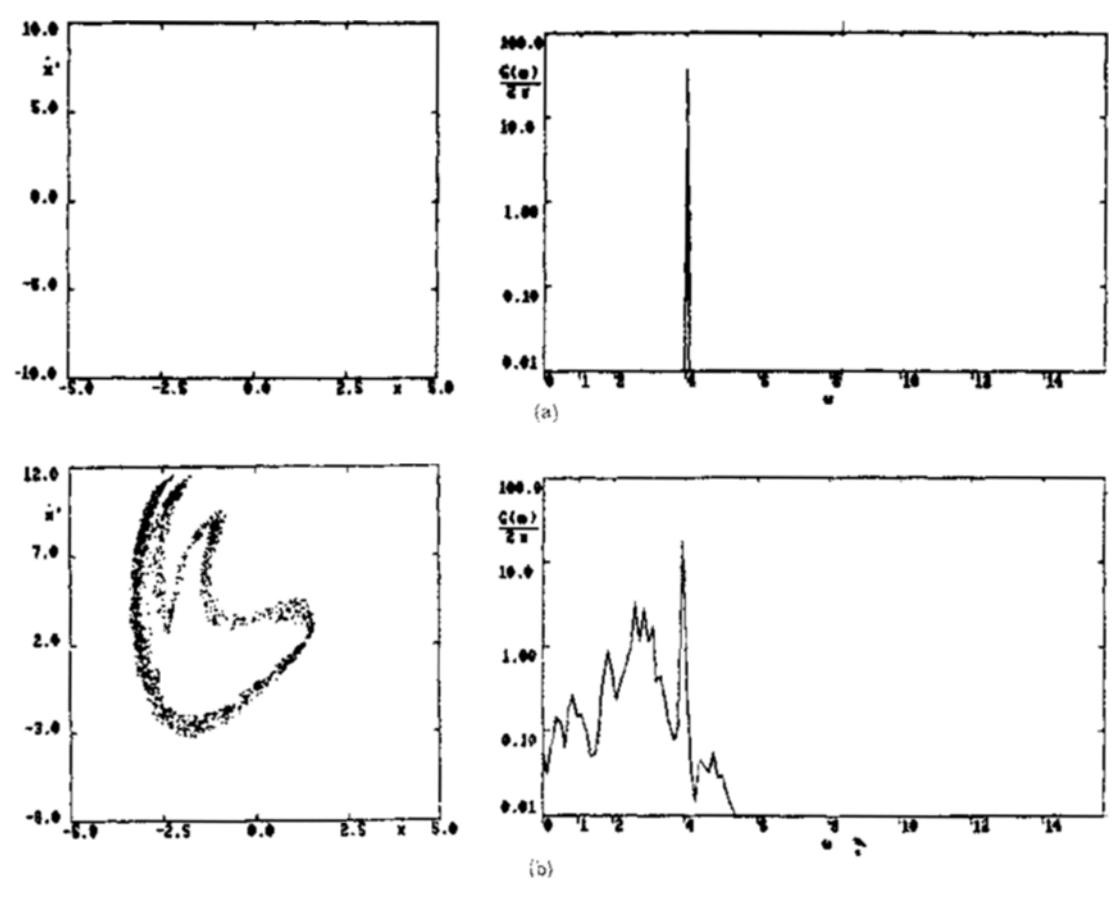


Fig. 4 Poincare maps (left) and frequency spectra (right) for $k=0.1$; ($T_{min}=39.3$, $T_{max}=3000.0$) ; a : $\tau_0=2.0$, b : $\tau_0=3.0$, c : $\tau_0=5.0$, d : $\tau_0=10.0$



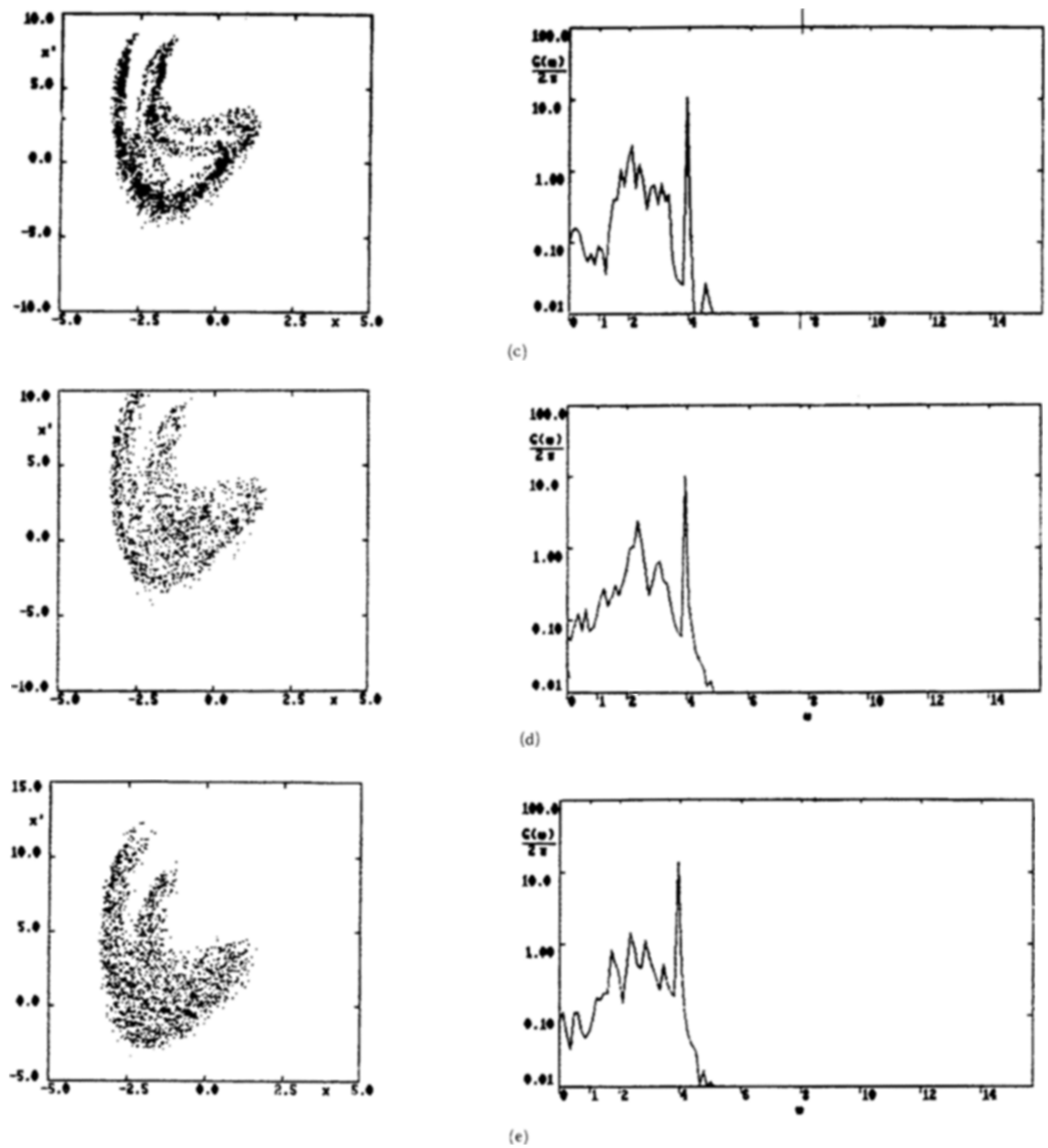


Fig. 5 Poincare maps (left) and frequency spectra (right) for $k=1.0$; ($T_{min}=39.3$, $T_{max}=3000.0$) ; a : $\tau_0=1.0$, b : $\tau_0=2.0$, c : $\tau_0=3.0$, d : $\tau_0=5.0$, e : $\tau_0=10$

maps are similar. The differencies are clearly visible in the shape of the broad-band Fourier spectra.

8. CONCLUDING REMARKS

We have investigated irregular motion in a nonlinear harmonically excited oscillator with delay. The influence of the delay unit parameters, ie. the amplification factor k and delay factor τ_0, has been examined considering them as perturbation parameters on the chaotic attractor analysed by Ueda and Akamatsu. We have shown that for some value of k and τ_0 only a nonchaotic attracting orbits occurs, whereas at some other value of the parameter, a chaotic attractor occurs. Based on the computer simulations we have demonstrated various types of regular and chaotic behaviour of the simple nonlinear oscillator. In addition to the chaotic attractor found by Ueda and Akamatsu at $k=0$, further chaotic attractors have been found for certain other values of the delay parameters. The route to chaos has been shown to be via successive cascades of frequency bifurcation, chaos being followed by a transition to regular motion. There are

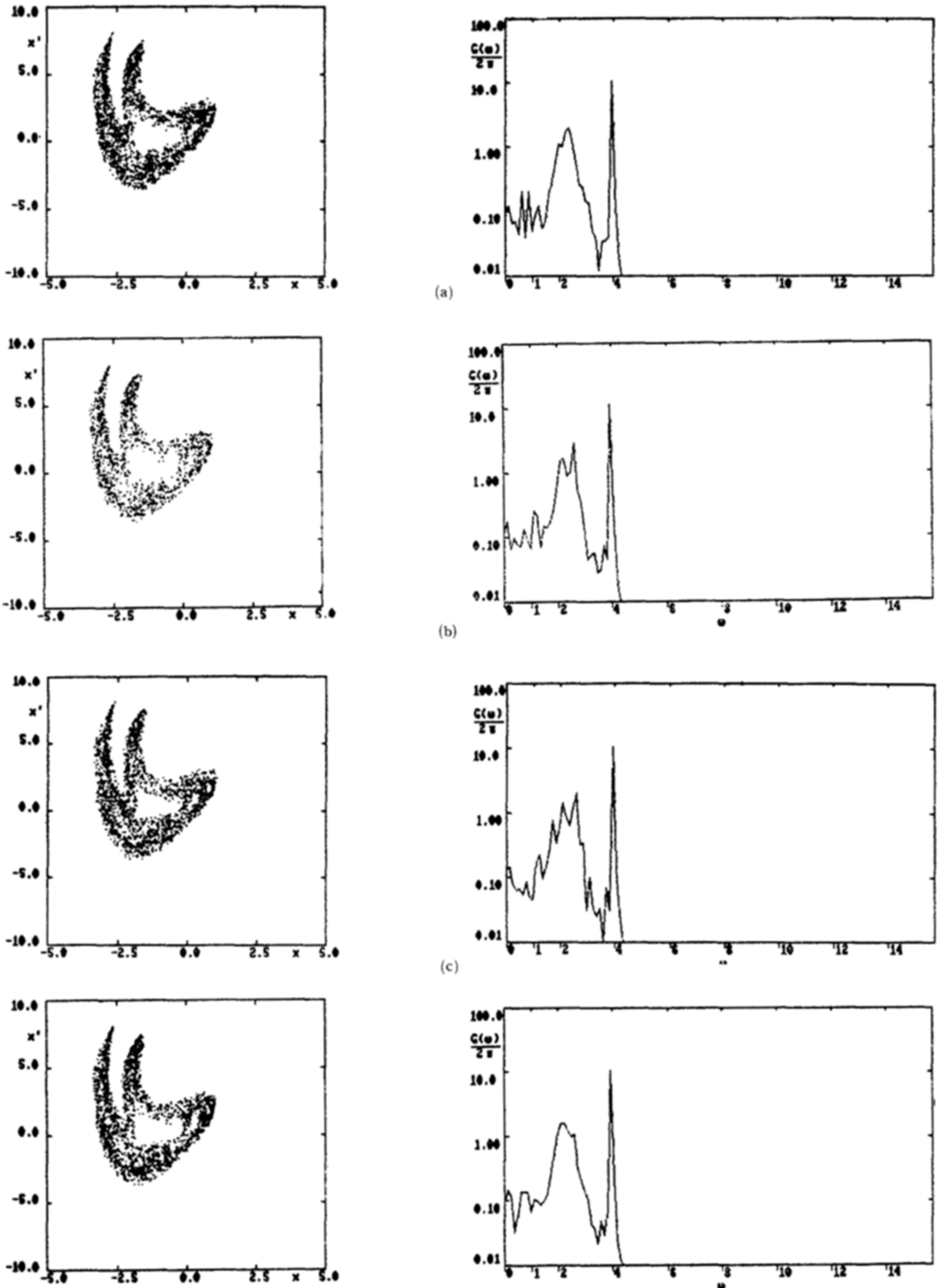


Fig. 6 Poincare maps (left) and frequency spectra (right) for $k=0.001$; ($T_{min}=78.5$, $T_{max}=4000.0$) ; a : $\tau_0=2.0$, b : $\tau_0=3.0$, c : $\tau_0=5.0$, d : $\tau_0=10.0$

intervals of k and τ_0 for which the motion of the oscillator is harmonic, with the frequency of the exciting force, or which is periodic with only two frequencies. We have also detected attractors which are non-chaotic, but the oscillator motion in this case is very complicated.

To distinguish the chaotic orbits we have used Poincare maps and Fourier spectra (FFT procedure). These two tools are very convenient for use during the investigation of real mechanical systems (see, for instance, Awrejcewicz and Barron, 1988).

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